6.5 Equations of Lines

Now that we have given a full treatment to finding the graph of a line when given its equation, we want to, in a sense, work that idea backwards.

That is to say, we want to be able to take information that we can gather from a graph, and produce the equation of the line that the graph represents.

To do this, first we need to know several different forms of the equation of a line.

Forms of the Equation of a Line

Slope-Intercept form:

$y = mx + b$ m is the slope, b is the y-intercept

Point-Slope form:

$y - y_1 = m(x - x_1)$ m is the slope, (x_1, y_1) is a point on the line

Standard form:

$Ax + By = C$ where A, B and C are integers

As it turns out, finding the equation is not terribly complicated. We are usually given enough, or can find enough information, to be able to use the Point-Slope form to initially get the equation.

Then from that point, we simply need to change the equation into whichever form we desire.

Example1:

Find the equation of the line in slope-intercept form using the given information.

a. (-3, 1), m = 2 b. (5, -1), $m = -\frac{1}{5}$ c. (3, 2), (4, 5) d. (-6, 2), (4, -3)

Solution:

a. To find the equation in slope-intercept form, we first notice that we are given a point and the slope. Since this is the case, it seems logical to use the point-slope form to generate our equation. So we begin with labeling the point (-3, 1) as (x_1, y_1). So $x_1 = -3$, $y_1 = 1$ and we already know that m = 2. Plugging them in we get

$$y - y_1 = m(x - x_1)$$
$$y - 1 = 2(x - (-3))$$

Now we just need to get it to slope intercept form by distributing and solving for y. This gives

$$y - 1 = 2\big(x - (-3)\big)$$

$$y - 1 = 2(x + 3) \qquad \text{Distribute to clear ( )}$$

$$\begin{aligned} y - 1 &= 2x + 6 \\ +1 \quad &\quad\ \ +1 \end{aligned} \qquad \text{Move the 1 over to get y alone}$$

$$y = 2x + 7$$

So the equation of the line which contains (-3, 1) and has slope of 2 is y = 2x + 7.

b. Just like above, we simply label (5, -1) as (x_1, y_1) and plug all the information into the point-slope form.

$$y - y_1 = m(x - x_1)$$

$$y - (-1) = -\frac{1}{5}(x - 5) \qquad \text{Distribute to clear ( )}$$

$$\begin{aligned} y + 1 &= -\frac{1}{5}x + 1 \qquad \text{Move the 1 over to get y alone} \\ -1 \quad & \qquad\quad -1 \\ y &= -\frac{1}{5}x \end{aligned}$$

So the equation of the line containing (5, -1) and has slope of $-\frac{1}{5}$ is $y = -\frac{1}{5}x$.

c. This time we are simply given two points (3, 2) and (4, 5). But as we can clearly see from parts a and b we need to slope to use the point-slope form.

Recall from section 6.3 that we have a formula for finding the slope when we are given two points. So all we need to do is label (3, 2) as (x_1, y_1) and (4, 5) as (x_2, y_2), then use the formula.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{5 - 2}{4 - 3} \\ &= \frac{3}{1} \\ &= 3 \end{aligned}$$

Now that we have the slope, we can use either point, and the slope, in the point-slope form to get our equation. To make it simple, let's use the point (3, 2) since it has already been labeled (x_1, y_1)

$$y - y_1 = m(x - x_1)$$

$$y - 2 = 3(x - 3) \qquad \text{Distribute to clear ( )}$$

$$\begin{aligned} y - 2 &= 3x - 9 \qquad \text{Move the 2 over to get y alone} \\ +2 \quad & \qquad +2 \\ y &= 3x - 7 \end{aligned}$$

So our equation is y = 3x – 7.

d. Last, we find the equation just like we did in part c above but now use the points (-6, 2) and (4, -3). Start by labeling (-6, 2) as (x_1, y_1) and (4, -3) as (x_2, y_2), and find the slope.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{-3-2}{4-(-6)}$$

$$= \frac{-5}{10}$$

$$= -\frac{1}{2}$$

Now use the point (-6, 2) and the point-slope form to get our line.

$$y - y_1 = m\,(x - x_1)$$

$$y - 2 = -\frac{1}{2}\big(x - (-6)\big) \qquad \text{Distribute to clear ( )}$$

$$y - 2 = -\frac{1}{2}x - 3 \qquad \text{Move the 2 over to get y alone}$$
$$+2 \qquad\qquad +2$$

$$y = -\frac{1}{2}x - 1$$

So our equation is y = - ½ x – 1.

Example 2:

Find the equation of the line in standard form using the given information.

a. (-2, 3), m = ½ b. (-1, 5), m = $-\frac{4}{7}$ c. (3, -2), (-3, -3) d. (-7, 5), (7, 1)

Solution:

a. The only difference between Example 2 and what we did in Example 1 is that here we want the ending equation to be in standard form. We work the process the same way, but in the end we want the equation to look like "variable terms = constant" with no fractions (since standard form requires A, B and C to be integers)

So we begin by labeling (-2, 3) as (x_1, y_1) and inserting everything into the point-slope form.

$$y - y_1 = m\,(x - x_1)$$
$$y - 3 = \frac{1}{2}\big(x - (-2)\big)$$

$$y - 3 = \frac{1}{2}(x + 2)$$

At this point, however, we need to start working twords standard form. So, since we need to clear the fractions by the end of the problem, its best to do this right after we distribute to clear the parenthesis. Then, we simply get all the variable terms to one side and the constant term to the other.

$$y - 3 = \frac{1}{2}(x + 2)$$ Distribute to clear ( )

$$y - 3 = \frac{1}{2}x + 1$$

$$2(y - 3) = 2\left(\frac{1}{2}x + 1\right)$$ Multiply by LCD of 2

$$\begin{aligned} 2y - 6 &= x + 2 \\ -2y \quad & \quad - 2y \end{aligned}$$ Move 2y over and flip-flop the equation

$$\begin{aligned} x - 2y + 2 &= -6 \\ -2 \quad & \quad - 2 \end{aligned}$$ Move the 2 over

$$x - 2y = -8$$

So the standard form equation is x – 2y = -8.

b. We proceed just as we did in part a. Start by labeling (-1, 5) as (x_1, y_1) and inserting everything into the point-slope form.

$$y - y_1 = m\,(x - x_1)$$

$$y - 5 = -\frac{4}{7}\big(x - (-1)\big)$$

$$y - 5 = -\frac{4}{7}(x + 1)$$

Now we just need to get it to standard form by clearing parenthesis, clearing fractions, and getting variable terms on one side, constant terms on the other.

$$y - y_1 = m\,(x - x_1)$$

$$y - 5 = -\frac{4}{7}(x + 1)$$

Distribute to clear ( )

$$y - 5 = -\frac{4}{7}x - \frac{4}{7}$$

$$7(y - 5) = 7\left(-\frac{4}{7}x - \frac{4}{7}\right)$$ Multiply by LCD of 7

$$\begin{aligned} 7y - 35 &= -4x - 4 \\ +4x \quad & \quad + 4x \end{aligned}$$ Move 4x over

$$\begin{aligned} 4x + 7y - 35 &= -4 \\ +35 \quad & \quad + 35 \end{aligned}$$ Move the 35 over

$$4x + 7y = 31$$

So the standard form equation is 4x + 7y = 31.

Notice in part a, we moved the variable terms to the right side, but in part b we moved the variable terms to the left side. The reason for this discrepancy is, as a general convention, we like to move the variable terms to whatever side will make the "x" term positive. Then if needed, we can flip the equation around to make the equation in the form Ax +By = C.

c. Recall that in Example 1, whenever we were given two points, we had to start with finding the slope of the points. It is the exact same idea here. Start by labeling (3, -2) as (x_1 , y_1) and (-3, -3) as (x_2 , y_2), and find the slope.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{-3 - (-2)}{-3 - 3}$$

$$= \frac{-1}{-6}$$

$$= \frac{1}{6}$$

Now use the point (3, -2) and the point-slope form to get our line.

$$y - y_1 = m\,(x - x_1)$$

$$y - (-2) = \tfrac{1}{6}(x - 3)$$ Distribute to clear ( )

$$y + 2 = \frac{1}{6}x - \frac{1}{2}$$

$$6(y + 2) = 6\left(\frac{1}{6}x - \frac{1}{2}\right)$$ Multiply by LCD of 6

$$\begin{aligned} 6y + 12 &= x - 3 \\ -6y \quad &\quad\; - 6y \end{aligned}$$ Move 6y over and flip-flop the equation

$$\begin{aligned} x - 6y - 3 &= 12 \\ +3 \quad &\quad + 3 \end{aligned}$$ Move the 3 over

$$x - 6y = 15$$

So the standard form equation is x – 6y = 15.

d. Finally, we find the equation as we did in part c. Label (-7, 5) as (x_1 , y_1) and (7, 1) as (x_2 , y_2), and find the slope

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{1 - 5}{7 - (-7)}$$

$$= \frac{-4}{14}$$

$$= -\frac{2}{7}$$

Now use the point (-7, 5) and the point-slope form to get our line.

$$y - y_1 = m\,(x - x_1)$$

$$y - 5 = -\tfrac{2}{7}\big(x - (-7)\big)$$ Distribute to clear ( )

$$y - 5 = -\frac{2}{7}x - 2$$

$$7(y - 5) = 7\left(-\frac{2}{7}x - 2\right)$$ Multiply by LCD of 7

$$\begin{array}{rcl} 7y - 35 & = & -2x - 14 \\ +2x & & +\,2x \end{array}$$ Move 2x over

$$\begin{array}{rcl} 2x + 7y - 35 & = & -14 \\ +35 & & +\,35 \end{array}$$ Move the 35 over

$$2x + 7y = 21$$

So the standard form equation is 2x + 7y = 21.

<u>Example 3</u>:

Find the equation of the line.

a. (3, -3), (4, -3)   b. (-1, 5), (-1, -6)

Solution:

a. As we did in the previous examples, let's start by finding the slope.
Label (3, -3) as (x_1 , y_1) and (4, -3) as (x_2 , y_2) and use the formula

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{-3 - (-3)}{4 - 3}$$

$$= \frac{0}{1}$$

$$= 0$$

Now, recall from section 6.3 that only horizontal lines have a slope that is zero. So we must have a horizontal line in this case. Also, recall from section 6.2 that every horizontal line has an equation of the form y = b.

Since both ordered pairs have a y value of -3, the equation must be y = -3.

b. Again, let's begin with finding the slope.
Label (-1, 5) as (x_1 , y_1) and (-1, -6) as (x_2 , y_2) and use the formula

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{-6-5}{-1-(-1)}$$

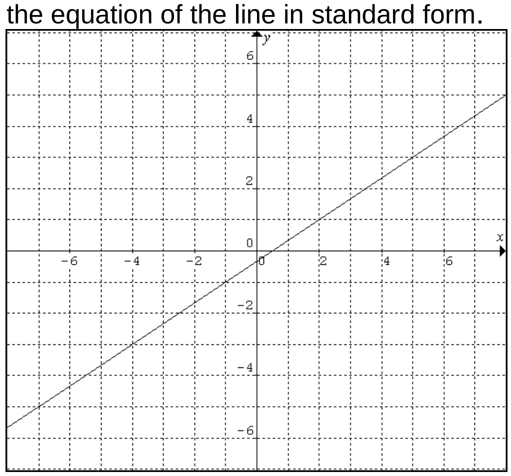
$$= \frac{-11}{0}$$
$$= undefined$$

Now, recall from section 6.3 that only vertical lines have a slope that is undefined. So we must have a vertical line in this case. Also, recall from section 6.2 that every vertical line has an equation of the form x = a.

Since both ordered pairs have an x value of -1, the equation must be x = -1.

<u>Example 4</u>:

Use the given graph to find the equation of the line in standard form.



Solution:

The first thing we need to do is get some information from the graph that we could use to make the equation, for example, 2 points or a point and the slope.

As it turns out, it's best to find two points that are on the graph and use the concept of slope to find the slope.

So in our example its clear to see that the line passes through the points (-1, -1) and (2, 1). So, the change in the y-direction from the first point to the second is +2 units and the change in the x-direction is +3 units. Therefore our slope must be $\frac{change\ in\ y}{change\ in\ x} = \frac{2}{3}$.

Now we can use the point (-1, -1) in our point-slope form to find the equation.

$$y - y_1 = m\,(x - x_1)$$

$$y - (-1) = \frac{2}{3}\big(x - (-1)\big)$$

$$y + 1 = \frac{2}{3}x + \frac{2}{3}$$

$$3(y+1)=3\left(\frac{2}{3}x+\frac{2}{3}\right)$$

$$\begin{aligned} 3y+3 &= 2x+2 \\ -3y \quad & \quad -3y \end{aligned}$$

$$\begin{aligned} 2x-3y+2 &= 3 \\ -2 \quad & \quad -2 \end{aligned}$$

$$2x-3y=1$$

So the standard form equation is 2x - 3y = 1.

6.5 Exercises

Find the equation of the line in slope-intercept form using the given information.

1. m = –2, (0, 3)
2. m = 3, (0, -1)
3. m = $-\frac{2}{3}$, (0, 3)
4. m = $-\frac{1}{7}$, (0, 0)
5. m = 3, (1, -3)
6. m = –2, (-2, 5)
7. m = ½ , (4, -3)
8. m = - 4, ($\frac{1}{2}$, 1)
9. m = -2, $\left(\frac{1}{4},-\frac{1}{2}\right)$
10. m = undef., (-5, 1)
11. m = 0, (-1, 2)
12. m = undef., (-2, 0)
13. $(5,1)$, $(4,3)$
14. $(4,5)$, $(2,9)$
15. $(6,-3)$, $(-3,0)$
16. $(2,5)$, $(2,3)$
17. $(-1,2)$, $(0,2)$
18. $\left(\frac{1}{2},\frac{1}{3}\right),\left(-\frac{1}{2},-\frac{2}{3}\right)$
19. $(-0.5,7.1),(-3,-1)$
20. $\left(3\frac{2}{3},5\frac{1}{2}\right)$, $\left(-1\frac{1}{3},4\frac{2}{5}\right)$

Find the equation of the line in standard form using the given information.

21. m = 4, (0, 0)
22. m = –1, (0, $-\frac{1}{3}$)
23. m = $-\frac{5}{2}$, (0, -5)
24. m = $-\frac{4}{3}$, (0, -2)
25. m = –4, (1, 5)
26. m = 3, (-2, 5)
27. m = $-\frac{2}{3}$, (-6, 5)
28. m = –3, ($-\frac{2}{3}$, 4)
29. m = 6, $\left(-\frac{2}{3},\frac{1}{3}\right)$
30. m = 0, (5, -2)
31. m = undef., (4, -2)
32. m = 0, (1, 3)
33. $(-3,-2)$, $(-2,2)$
34. $(3,-1)$, $(6,-2)$
35. $(2,7)$, $(4,-2)$
36. $(1,1)$, $(4,0)$
37. $(-3,-2)$, $(5,-2)$
38. $(-3,7)$, $(-3,-1)$
39. $(10,0)$, $(0,-10)$
40. (1.2, -3.4), (-0.8, -5)
41. $(0,0)$, $(1,1)$
42. $(5,-2),(-3,7)$
43. $(6,-2)$, $(2,7)$
44. $(2,9)$, $(6,-3)$
45. $(-3,0)$, $(2,5)$
46. (1, -3), (-2, 5)
47. (4, -3), ($\frac{1}{2}$, 1)
48. $(-2,2)$, $(3,-1)$
49. $\left(\frac{5}{6},\frac{1}{4}\right)$, $\left(\frac{2}{3},\frac{1}{2}\right)$
50. $\left(1\frac{1}{3},2\frac{1}{5}\right),\left(2\frac{3}{5},-7\frac{1}{3}\right)$

51. Find the equation of the x-axis.

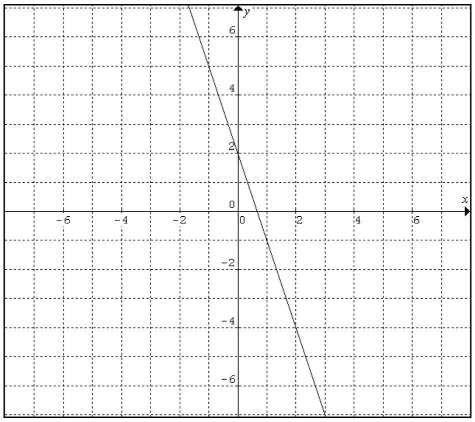
52. Find the equation of the y-axis.

53. Find the equation of the line with slope 2π and passing through the point $(0,0)$.

54. Find the equation of the line with slope $(\pi + 4)$ and passing through the point $(0,0)$.

Find the equation of the line in the given graph.

55.



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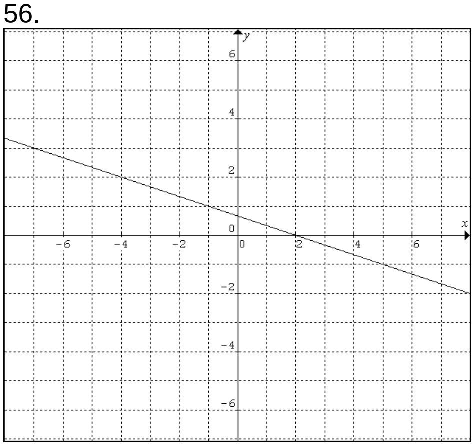

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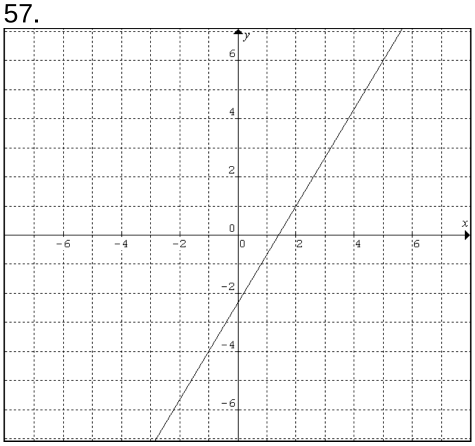

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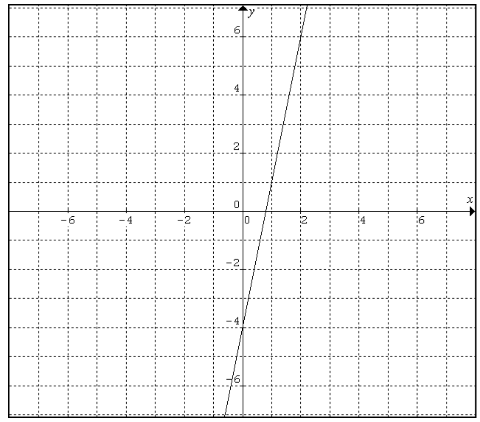

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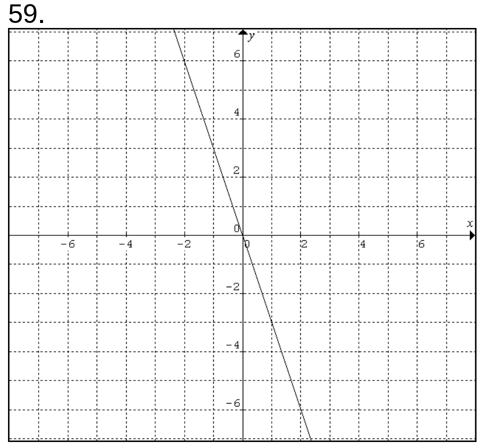

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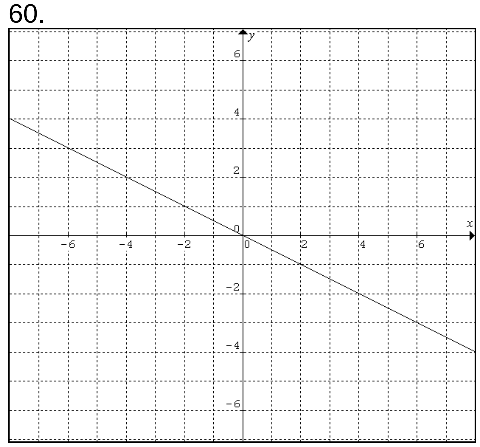

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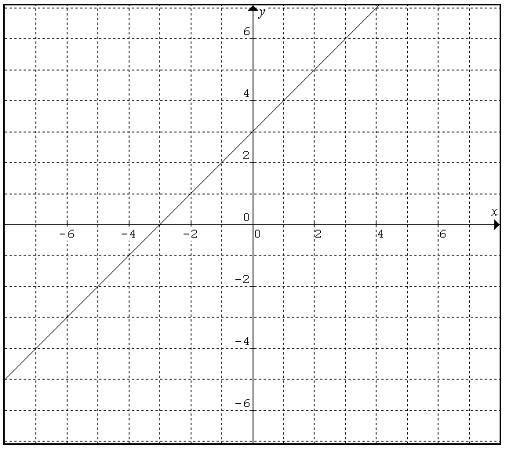

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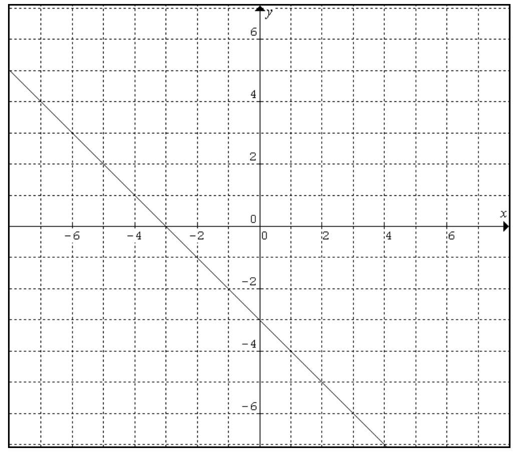

Three points are called **collinear** if they are all on the same line. Determine if the three given points are collinear.

63. (2, -2), (3, 4), (-1, 5)
64. (-3, 7), (0, 2), (1, -5)
65. (1, -3), (2, 1), (3, 5)
66. (1, 1), (-3, -6), (3, 2)