9.6 Inverse Functions

We would now like to turn our attention to a specific family of functions, the one to one functions.

Definition: One to One function
A function f is called 1-1 if, for any a and b in the domain of f, $f(a) = f(b)$ implies that $a = b$.

The one to one idea is similar to that of the basic function idea. Whereas, the basic function idea is that for every one domain value, you only have one range value, for a one to one function, the idea is every one range value only comes from one domain value. That is, the function cannot repeat any y-value to be 1-1.

<u>Example 1</u>: Show that the function $f(x) = \frac{1}{3}x + 2$ is 1-1.

Solution:

We need to start with $f(a) = f(b)$ and show that $a = b$. So, since $f(a) = \frac{1}{3}a + 2$ and $f(b) = \frac{1}{3}b + 2$ we get

$$f(a) = f(b)$$
$$\tfrac{1}{3}a + 2 = \tfrac{1}{3}b + 2 \qquad \text{Subtract 2 from both sides}$$
$$\tfrac{1}{3}a = \tfrac{1}{3}b \qquad \text{Multiply 3 on both sides}$$
$$a = b$$

So $f(x) = \frac{1}{3}x + 2$ is 1-1.

Since the 1-1 idea is so similar to the function idea, we have a test similar to the vertical line test to test if a function is 1-1.

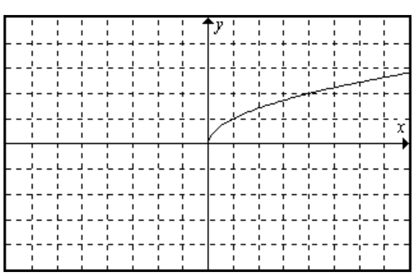
The Horizontal Line Test
The graph of a function represents the graph of a 1-1 function if any horizontal line intersects the graph at no more than one point.

So to test a graph of a function to see if it is 1-1, we simply need to see if any horizontal line intersects more than one point.

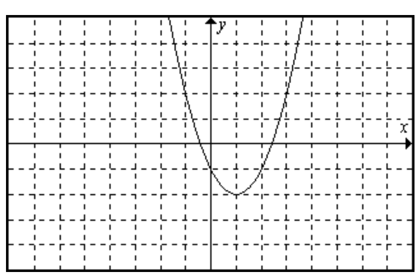
<u>Example 2</u>:

Determine which of the following functions are 1-1

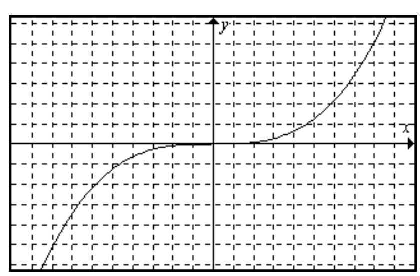
a.



b.



c.



Solution:

Clearly parts a. and c. pass the horizontal line test and part b. does not. Therefore, a. and c. are one to one functions and b. is not.

Although one to one functions are important, there main application comes in the way of the inverse function, which is our next definition.

Definition: Inverse function
Let f and g be two functions such that $f(g(x)) = x$ for all *x* in the domain of g and $g(f(x)) = x$ for all *x* in the domain of f, then the function g is called the inverse function of f, written f^{-1}.

For the sake of simplification, we will not worry about the domains involved. Readers curious about this can read about inverse functions in a precalculus textbook.

The idea of the inverse function is that two functions are inverses if the composition in both directions always gives you *x*.

<u>Example 3</u>:

Verify that the functions $f(x) = 2x - 1$ and $g(x) = \frac{1}{2}(x+1)$ are inverse functions.

Solution:

In order to verify that the functions are inverses we simply need to show that $f(g(x)) = x$ and $g(f(x)) = x$. We will start with $f(g(x)) = x$.

$$\begin{aligned} f(g(x)) &= f\left(\tfrac{1}{2}(x+1)\right) \\ &= 2\left(\tfrac{1}{2}(x+1)\right) - 1 \\ &= (x+1) - 1 \\ &= x \end{aligned}$$

So $f(g(x)) = x$. Now we show $g(f(x)) = x$.

$$\begin{aligned} g(f(x)) &= g(2x-1) \\ &= \tfrac{1}{2}((2x-1)+1) \\ &= \tfrac{1}{2}(2x) \\ &= x \end{aligned}$$

So since $f(g(x)) = x$ and $g(f(x)) = x$, the functions are inverse functions. We write g as $f^{-1}(x) = \frac{1}{2}(x+1)$.

As a consequence of the above definition we clearly get the following property.

Property of Inverse Functions
$f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$

We don't want to simply be able to show that two functions are inverse functions. We would actually like to start with a given function and find what the inverse function is, that is, if it exists. We have the following steps to finding the inverse function.

Finding an Inverse Function
1. In the equation for $f(x)$, replace $f(x)$ with y. 2. Interchange *x* and *y.* 3. Solve the resulting equation for *y*. 4. Replace *y* by $f^{-1}(x)$.

<u>Example 4</u>:

Let $f(x)=6x+1$. Find $f^{-1}(x)$.

Solution:

First we replace $f(x)$ with y to get $y=6x+1$.
Now we interchange *x* and *y* to get $x=6y+1$.
Next we solve the equation for *y*.

$$x=6y+1$$
$$x-1=6y$$
$$y=\frac{x-1}{6}$$

Finally, we replace *y* by $f^{-1}(x)$ to get $f^{-1}(x)=\frac{x-1}{6}$.

<u>Example 5</u>:

Let $f(x)=\frac{4}{x-2}$. Find $f^{-1}(x)$.

Solution:

Again we will follow the steps outlined above.

$f(x)=\dfrac{4}{x-2}$	Replace $f(x)$ with y
$y=\dfrac{4}{x-2}$	Interchange *x* and *y*
$x=\dfrac{4}{y-2}$	Solve for *y*
$(y-2)x=4$	Multiply by the LCD $y-2$
$y-2=\dfrac{4}{x}$	Divide by *x*
$y=\dfrac{4}{x}+2$	
$f^{-1}(x)=\dfrac{4}{x}+2$	Replace *y* by $f^{-1}(x)$

<u>Example 6</u>:

Let $f(x)=\frac{1}{2}\sqrt[3]{x+1}$. Find $f^{-1}(x)$.

Solution:

$f(x)=\frac{1}{2}\sqrt[3]{x+1}$	Replace $f(x)$ with y
$y=\frac{1}{2}\sqrt[3]{x+1}$	Interchange *x* and *y*
$x=\frac{1}{2}\sqrt[3]{y+1}$	Solve for *y*
$2x=\sqrt[3]{y+1}$	
$(2x)^3=y+1$	Cube both sides to eliminate the radical
$y=8x^3-1$	
$f^{-1}(x)=8x^3-1$	Replace *y* by $f^{-1}(x)$

We can see from the last example that powers and roots are inverses of one another. So, when trying to find an inverse of an equation that contains exponents we take a radical of both sides and vice versa.

Notice by the way we find an inverse function, that since we interchange the roles of *x* and *y*, it would make sense that the domains and ranges would interchange as well, since they are associated with the *x* and *y* This is exactly the case. We state this as the following fact.

Fact: The domain of f is the range of f^{-1} and the range of f is the domain of f^{-1}. Or alternately, the domain of f^{-1} is the range of f and the range of f^{-1} is the domain of f.

Finally we would like to talk about the geometry of a function and its inverse function.

We start with the following condition.

Condition for an Inverse Function

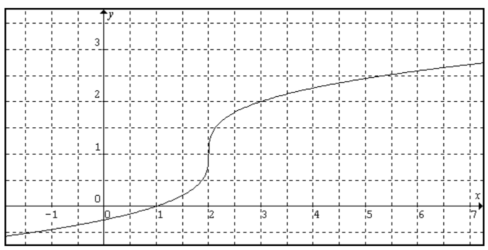
A function has an inverse function if and only if it is 1-1

What that means for us is that a function has an inverse function only when it passes the horizontal line test.

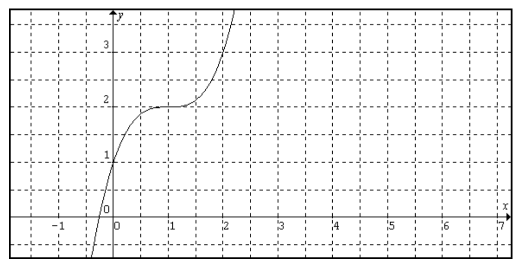
More importantly though is the relationship between the graphs of a function and its inverse function.

Since in order to find an inverse function we interchange the roles of *x* and *y*, then it would make sense that if the point (a, b) is on the graph of a function then (b, a) would have to be on the inverse function. This relationship corresponds to a reflection across the line $y = x$.

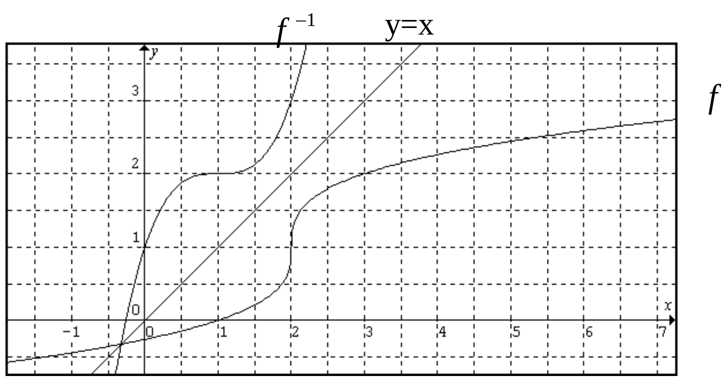
So if the graph of f is



Then the graph of f^{-1} would be



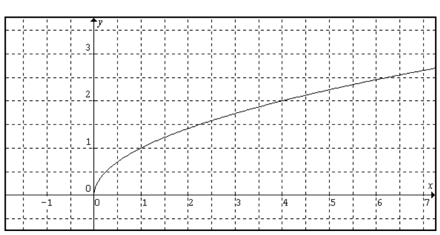
If we graph these on the same axis along with the line $y = x$ we can really see the reflection.



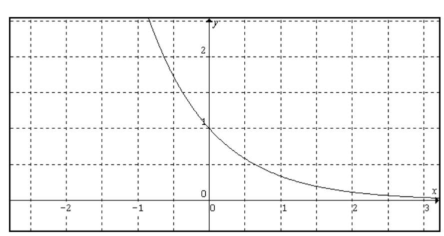
Example 7:

Given the graph of f, graph f^{-1}.

a.



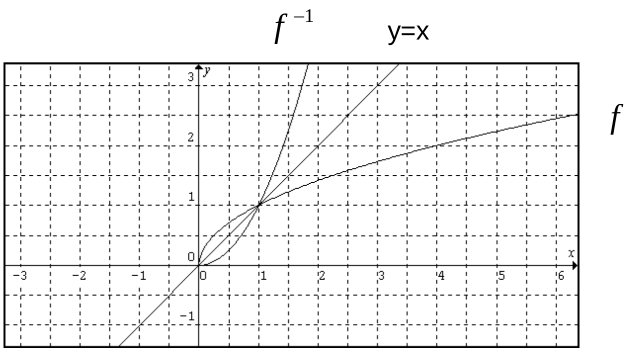
b.



Solution:

We simply need to reflect each graph over the line $y = x$.

a.



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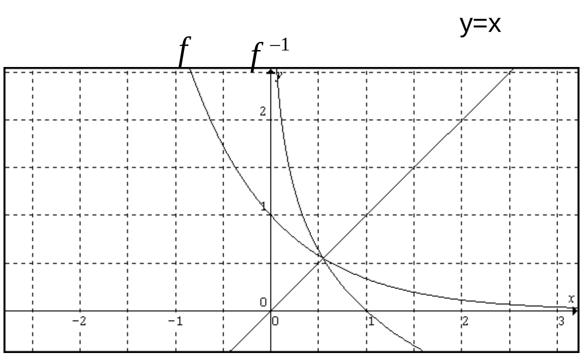


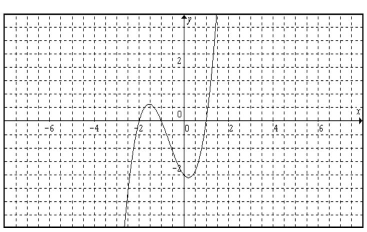
9.6 Exercises

Show that the following functions are 1-1.

1. $f(x)=7x+4$
2. $f(x)=4x+9$
3. $f(x)=\frac{1}{3}x+\frac{1}{4}$
4. $f(x)=-\frac{1}{2}x+3$
5. $g(x)=\sqrt{x-1}$
6. $g(x)=\sqrt{2-x}$
7. $g(x)=2\sqrt[3]{x}$
8. $g(x)=\dfrac{1}{x}$
9. $h(x)=\dfrac{1}{x+1}$
10. $h(x)=\dfrac{2}{x}-3$

Determine if the graph represents a 1-1 function.

11.



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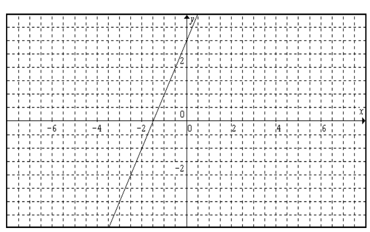

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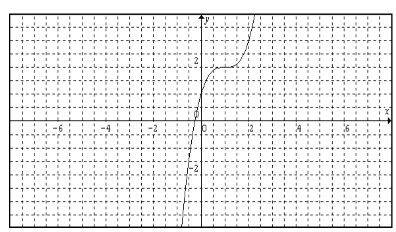

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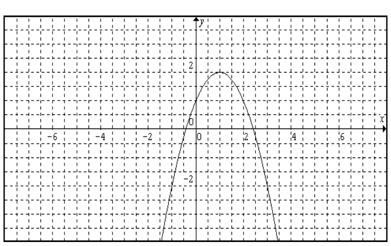

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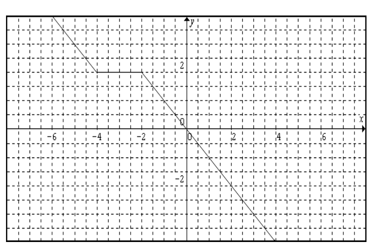


Verify that the two functions are inverse functions.

16. $f(x)=2x-5,\ g(x)=\frac{x+5}{2}$

17. $f(x)=x^7,\ g(x)=\sqrt[7]{x}$

18. $f(x)=x^{1/5},\ g(x)=x^5$

19. $f(x)=3+\frac{1}{x},\ g(x)=\frac{1}{x-3}$

20. $f(x)=\sqrt[3]{x+1},\ g(x)=x^3-1$

21. $f(t)=t^3-1,\ g(t)=\sqrt[3]{t+1}$

22. $f(u)=\frac{4}{3-u},\ g(u)=3-\frac{4}{u}$

23. $f(x)=\frac{1}{16}x^4,\ g(x)=2\sqrt[4]{x}$

24. $f(x)=(x+2)^5,\ g(x)=\sqrt[5]{x}-2$

25. $f(x)=x^3+1,\ g(x)=\sqrt[3]{x-1}$

26. $f(n)=\frac{1}{1+n},\ g(n)=\frac{1}{n}-1$

27. $f(x)=\frac{4}{x-2}+3,\ g(x)=\frac{4}{x-3}+2$

28. $f(x)=\frac{5}{3x-9},\ g(x)=\frac{5}{3x}+3$

29. $f(x)=\frac{2}{x+1}-1,\ g(x)=\frac{2}{x+1}-1$

30. $f(x)=4-\sqrt[15]{x+1},\ g(x)=(4-x)^{15}-1$

Find the inverse of the following.

31. $f(x)=7x+4$

32. $f(x)=4x+9$

33. $f(x)=\frac{1}{3}x+\frac{1}{4}$

34. $f(x)=-\frac{1}{2}x+3$

35. $f(x)=\frac{1}{2}-\frac{1}{4}x$

36. $g(x)=\frac{1}{3}+\frac{1}{9}x$

37. $g(x)=\sqrt{x}$

38. $g(x)=2\sqrt[3]{x}$

39. $g(x)=\sqrt[3]{2-x}$

40. $g(x)=\sqrt{x+1}$

41. $g(x)=x^3+4$

42. $g(x)=-x^3-2$

43. $g(x)=2x^2+4$

44. $h(x)=1-x^2$

45. $h(x)=\frac{1}{x}$

46. $h(x)=\frac{2}{x}-3$

47. $h(x)=\frac{1}{x+1}$

48. $h(x)=\frac{1}{2x-3}$

49. $h(x)=\frac{1}{x-1}+3$

50. $f(x)=\frac{1}{2}\sqrt{x-2}+1$

51. $f(x)=\sqrt[3]{2x-3}+2$

52. $f(x)=\frac{x}{x+1}$

53. $f(x)=\frac{2x}{x-2}$

54. $g(x)=\frac{3x+1}{2-x}$

55. $g(x)=\frac{x-1}{x+1}$

56. $g(x)=2-2x^3$

57. $h(x)=7x^4+8$

58. $g(x)=\sqrt[3]{x-3}-2$

59. $f(x)=-2\sqrt{x+1}-1$

60. $g(x)=\sqrt{x^2+1}$

61. $f(x)=\sqrt[3]{x^3-2}$

62. $h(x)=\frac{3x-1}{2x-1}+1$

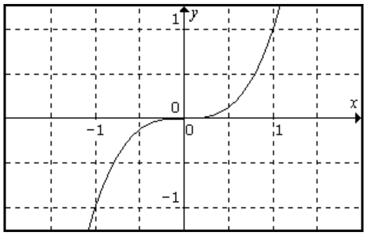
63. $f(x)=\frac{1}{x^2}-2$

64. $g(x)=\frac{4}{x^2+1}$

65. $h(x)=\frac{1}{\sqrt[3]{x}}+2$

Given the graph of f, graph f^{-1}.

66.



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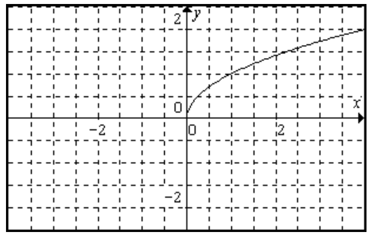

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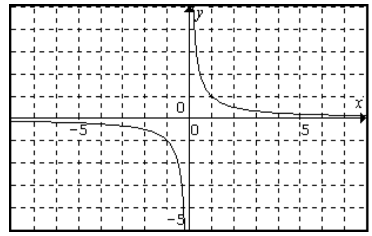

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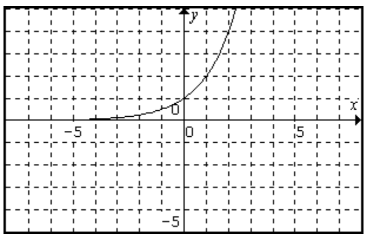

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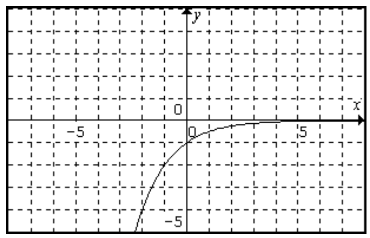

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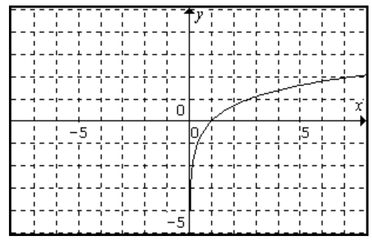

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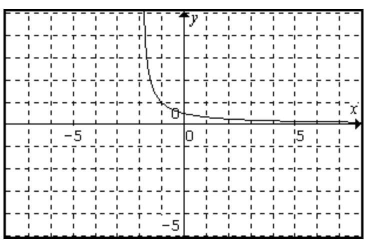

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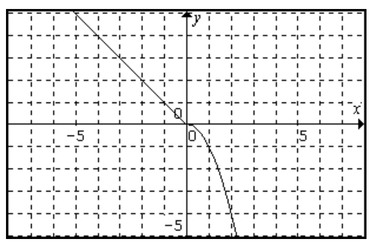

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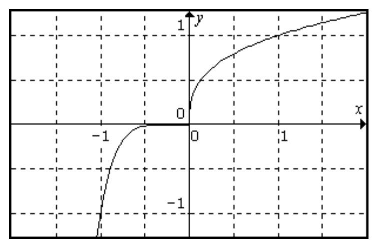

In Exercises 31-65 we found the inverse of a function using the techniques we learned in this section. However, sometimes these techniques fail us. For example, if $f(x)=x^2$ then we can calculate f^{-1} to get $f^{-1}(x)=\sqrt{x}$. But, $f(x)=x^2$ is not 1-1 and therefore cannot have an inverse function. However, if we change the domain to make it a 1-1 function, then it can have an inverse function. We simply change the domain of f to be $x \geq 0$. Similarly, if we started with $f(x)=\sqrt{x}$ we change the domain of $f^{-1}(x)=x^2$ to $x \geq 0$ thereby making it 1-1.

For the following exercises, find f^{-1}. Be sure to consider all domain stipulations.

State the domain and range of each f and f^{-1}.

75. $f(x)=\sqrt{x}+1$

76. $f(x)=\sqrt{x}-2$

77. $f(x)=\sqrt{x+1}$

78. $f(x)=\sqrt[4]{x-2}$

79. $f(x)=\sqrt[4]{2-x}$

80. $f(x)=\sqrt[4]{x-1}-1$

81. $f(x)=x^2+1,\ x \geq 0$

82. $f(x)=(x-1)^2,\ x \geq 1$

83. $f(x)=(x+2)^2,\ x \leq -2$

84. $f(x)=x^2+2,\ x \leq 0$

Let $f(x)=\dfrac{x}{x+1}$. Find the following.

85. $f^{-1}(x)$

86. $f^{-1}(-1)$

87. $f^{-1}(2)$

88. $f^{-1}(a)$

89. $f^{-1}(x+h)$

90. $\dfrac{f^{-1}(x+h)+f^{-1}(x)}{h}$